

# Orthogonality of linear (alinear) quasigroups and their parastrophes

V.A. Shcherbacov

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## Abstract

Necessary and sufficient conditions of orthogonality of left (right) linear (alinear) quasigroups in various combinations are given. As corollary we obtain conditions of parastroph orthogonality of left (right) linear (alinear) quasigroups. Any linear (alinear) quasigroup over the group  $S_n$  ( $n \neq 2; 6$ ) is not orthogonal to its (12)-parastrophe.

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## Contents

1	Introduction	1
2	Parastrophes of left(right) linear (alinear) quasigroups	5
3	Orthogonality of left (right) linear (alinear) quasigroups	7
4	Orthogonality of parastrophes of left(right) linear(alinear) quasigroups	17
4	References	22

## 1 Introduction

In [5] complete  $k$ -recursive MDS-codes are constructed using the systems of  $n$ -ary ( $n \geq 2$ ) orthogonal quasigroups. Systems of orthogonal  $n$ -ary operations ( $n \geq 2$ ) are used by construction of some crypto-algorithms [4, 13]. Therefore the study of quasigroup orthogonality is important from theoretical and "practical" point of view.

In introduction we give some basic definition. For more detailed information on basic concepts used in the paper it is possible to see [1, 3, 11, 12].

**Definition 1.** A binary groupoid  $(Q, A)$  with binary operation  $A$  such that in the equality  $A(x_1, x_2) = x_3$  knowledge of any two the elements  $x_1, x_2, x_3$  uniquely specifies the remaining one is called a binary quasigroup [2].

**Definition 2.** From Definition 1 it follows that with a given binary quasigroup  $(Q, A)$  it is possible to associate  $(3! - 1)$  others, so-called parastrophes of quasigroup  $(Q, A)$ :

$$\begin{aligned} A(x_1, x_2) = x_3 &\Leftrightarrow \\ A^{(12)}(x_2, x_1) = x_3 &\Leftrightarrow \\ A^{(13)}(x_3, x_2) = x_1 &\Leftrightarrow \\ A^{(23)}(x_1, x_3) = x_2 &\Leftrightarrow \\ A^{(123)}(x_2, x_3) = x_1 &\Leftrightarrow \\ A^{(132)}(x_3, x_1) = x_2. & \end{aligned}$$

[16, p. 230], [1, p. 18].

Notice, cases 5 and 6 are "(12)-parastrophes" of cases 3 and 4, respectively.

**Remark 3.** Sometimes the following definition of parastrophy is more convenient: all the same as in Definition 2 for exception of the last two cases  $A^{(123)}(x_3, x_1) = x_2 \Leftrightarrow A^{(132)}(x_2, x_3) = x_1$ , i.e., cases 5 and 6 are "(12)-parastrophes" of cases 4 and 3, respectively.

Here we follow tradition.

**Definition 4.** Let  $(Q, +)$  be a quasigroup. A permutation  $\overline{\varphi}$  of the set  $Q$  is called an anti-automorphism of quasigroup  $(Q, +)$ , if the following equality is true for all  $x, y \in Q$ :  $\overline{\varphi}(x + y) = \overline{\varphi}y + \overline{\varphi}x$ .

Denote by  $Aaut(Q, +)$  the set of all anti-automorphisms of a quasigroup  $(Q, +)$ .

**Definition 5.** A quasigroup  $(Q, \cdot)$  is called left linear, if  $x \cdot y = \varphi x + a + \beta y$ , where  $(Q, +)$  is a group, the element  $a$  is a fixed element of the set  $Q$ ,  $\varphi \in Aut(Q, +)$ ,  $\beta \in S_Q$ .

**Lemma 6.** If a left linear quasigroup  $(Q, \cdot)$  has the form  $x \cdot y = \varphi x + a + \beta y$  over a group  $(Q, +)$ , then it also has the form  $x \cdot y = \varphi x + J_a \beta y + a$  over the group  $(Q, +)$ , where  $J_a x = a + x - a$ , and vice versa.

*Proof.* Indeed, from equality  $x \cdot y = \varphi x + a + \beta y$  we have  $x \cdot y = \varphi x + a + \beta y - a + a = \varphi x + (a + \beta y - a) + a = \varphi x + J_a \beta y + a$ . Notice, the map  $J_a$  is an inner automorphism of the group  $(Q, \cdot)$  [6]. It is clear that the map  $J_a$  is a permutation of the set  $Q$ .  $\square$

**Definition 7.** A quasigroup  $(Q, \cdot)$  is called left alinear, if  $x \cdot y = \overline{\varphi}x + a + \beta y$ , where  $(Q, +)$  is a group, the element  $a$  is a fixed element of the set  $Q$ ,  $\overline{\varphi}$  is an anti-automorphism,  $\beta \in S_Q$ .

**Lemma 8.** If a left alinear quasigroup  $(Q, \cdot)$  has the form  $x \cdot y = \overline{\varphi}x + a + \beta y$  over a group  $(Q, +)$ , then it also has the form  $x \cdot y = \overline{\varphi}x + J_a \beta y + a$  over the group  $(Q, +)$ , where  $J_a x = a + x - a$ , and vice versa.

*Proof.* The proof is similar to the proof of Lemma 6.  $\square$

In [14] left (right) linear quasigroups over an abelian group are called *semicentral*.

**Definition 9.** 1. A quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \varphi x + a + \psi y$ , where  $(Q, +)$  is a group, the element  $a$  is a fixed element of the set  $Q$ ,  $\varphi, \psi \in Aut(Q, +)$ , is called linear quasigroup (over the group  $(Q, +)$ ).

2. A quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \varphi x + a + \overline{\psi}y$ , where  $(Q, +)$  is a group, the element  $a$  is a fixed element of the set  $Q$ ,  $\varphi \in \text{Aut}(Q, +)$ ,  $\overline{\psi} \in \text{Aaut}(Q, +)$ , is called left linear right alinear quasigroup (over the group  $(Q, +)$ ).
3. A quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \overline{\varphi}x + a + \psi y$ , where  $(Q, +)$  is a group, the element  $a$  is a fixed element of the set  $Q$ ,  $\psi \in \text{Aut}(Q, +)$ ,  $\overline{\varphi} \in \text{Aaut}(Q, +)$ , is called left alinear right linear quasigroup (over the group  $(Q, +)$ ).
4. A quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \overline{\varphi}x + a + \overline{\psi}y$ , where  $(Q, +)$  is a group, the element  $a$  is a fixed element of the set  $Q$ ,  $\overline{\varphi}, \overline{\psi} \in \text{Aaut}(Q, +)$ , is called alinear quasigroup (over the group  $(Q, +)$ ).

**Definition 10.** A quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \varphi x + \psi y + a$ , where  $(Q, +)$  is an abelian group, the element  $a$  is a fixed element of the set  $Q$ ,  $\varphi, \psi \in \text{Aut}(Q, +)$ , is called T-quasigroup [10, 7].

In Definition 9 we follow tradition but below, using Lemma 6, we pass to other form of linear (alinear) quasigroups more usable in "abelian" case, i.e., more usable by the study of  $T$ -quasigroups.

**Remark 11.** From Lemma 6 it follows:

1. a linear quasigroup  $(Q, \cdot)$  has the form  $x \cdot y = \varphi x + a + \psi y$  if and only if it has the following form:  $x \cdot y = \varphi x + I_a \psi y + a$ .
2. a left linear right alinear quasigroup  $(Q, \cdot)$  has the form  $x \cdot y = \varphi x + a + \overline{\psi}y$  if and only if it has the form  $x \cdot y = \varphi x + I_a \overline{\psi}y + a$ .
3. a left alinear right linear quasigroup  $(Q, \cdot)$  has the form  $x \cdot y = \overline{\varphi}x + a + \psi y$  if and only if it has the form  $x \cdot y = \overline{\varphi}x + I_a \psi y + a$ .
4. an alinear quasigroup  $(Q, \cdot)$  has the form  $x \cdot y = \overline{\varphi}x + a + \overline{\psi}y$  if and only if it has the form  $x \cdot y = \overline{\varphi}x + I_a \overline{\psi}y + a$ .

We give some elementary properties of quasigroup (group) automorphism and anti-automorphisms.

**Lemma 12.** 1. The product of two anti-automorphisms of a quasigroup  $(Q, +)$ , say  $\overline{\varphi}$  and  $\overline{\psi}$ , is an automorphism of  $(Q, +)$ .

2. If  $\varphi$  is an automorphism of a quasigroup  $(Q, +)$  and  $\overline{\psi}$  is its anti-automorphism, then  $\varphi \overline{\psi}$ ,  $\overline{\psi} \varphi$  are some anti-automorphisms of quasigroup  $(Q, +)$ .
3. Denote by the letter  $I$  the following anti-automorphism of a group  $(Q, +)$ :  $I(x) = -x$  for any  $x \in Q$ . It is well known that  $I^2 = \varepsilon$  [6]. Any anti-automorphism  $\overline{\psi}$  of a group  $(Q, +)$  can be presented in the form  $\overline{\psi} = I\psi$ , where  $\psi \in \text{Aut}(Q, +)$ .
4.  $I\varphi = \varphi I$ .
5.  $I\overline{\varphi} = \overline{\varphi} I$ .
6.  $(\overline{\varphi})^{-1} = \overline{\varphi^{-1}} = I\varphi^{-1}$ .
7. By  $J_a$  we denote inner automorphism of a group  $(Q, +)$ , i.e.  $J_a x = a + x - a$  for any  $x \in Q$ . If  $\varphi \in \text{Aut}(Q, +)$ , then  $\varphi J_a = J_{\varphi a} \varphi$ ,  $\varphi J_a^{-1} = J_{\varphi a}^{-1} \varphi$ .

8. If  $J_a \in \text{Inn}(Q, +)$  and  $I\varphi \in \text{Aut}(Q, +)$ , then  $I\varphi J_a = J_{\varphi a} I\varphi$ , i.e.,  $\overline{\varphi} J_a = J_{\varphi a} \overline{\varphi}$ .

9.  $IJ_a = J_a I$ .

*Proof.* 1. Indeed,  $\overline{\varphi}\overline{\psi}(x+y) = \overline{\varphi}(\overline{\psi}y + \overline{\psi}x) = \overline{\varphi}\overline{\psi}x + \overline{\varphi}\overline{\psi}y$ .

2. This is easy to check.

3. From the last equality we have the following  $I\overline{\psi} = \psi$ , that proves this statement.

4. We have  $0 = \varphi I(x + Ix) = \varphi(x + Ix) = \varphi x + \varphi Ix$ . From other side  $0 = \varphi x + I\varphi x$ . Therefore  $\varphi I = I\varphi$ .

5. We have  $\overline{\varphi} I = I\varphi I = \varphi$ ,  $I\overline{\varphi} = I^2\varphi = \varphi$ . Therefore  $\overline{\varphi} I = I\overline{\varphi}$ .

6.  $(\overline{\varphi})^{-1} = (I\varphi)^{-1} = \varphi^{-1} I = I\varphi^{-1} = \overline{\varphi^{-1}}$ .

7. We have  $\varphi J_a x = \varphi(a + x - a) = \varphi a + \varphi x - \varphi a = J_{\varphi a} \varphi$ .

8. We have  $\overline{\varphi} J_a x = I\varphi J_a x = I\varphi(a + x - a) = I(\varphi a + \varphi x - \varphi a) = -I\varphi a + I\varphi x + I\varphi a = \varphi a + I\varphi x - \varphi a = J_{\varphi a} I\varphi x = J_{\varphi a} \overline{\varphi} x$ .

9. Indeed,  $IJ_a x = I(a + x - a) = I(-a) + Ix + Ia = a + Ix - a = J_a Ix$ .

□

Below we shall use properties described in Lemma 12 without additional comments.

It is clear that any alinear quasigroup over an abelian group is linear since in any abelian group any antiautomorphism is an automorphism.

**Lemma 13.** 1. For any left linear quasigroup  $(Q, \cdot)$  there exists its form such that  $x \cdot y = \varphi x + \beta y$ .

2. For any right linear quasigroup  $(Q, \cdot)$  there exists its form such that  $x \cdot y = \alpha x + \psi y$ .

3. For any left alinear quasigroup  $(Q, \cdot)$  there exists its form such that  $x \cdot y = \overline{\varphi} x + \beta y$ .

4. For any right linear quasigroup  $(Q, \cdot)$  there exists its form such that  $x \cdot y = \alpha x + \overline{\psi} y$ .

*Proof.* 1. We can re-write the form  $x \cdot y = \varphi x + \beta y + c$  of a left linear quasigroup  $(Q, \cdot)$  as follows  $x \cdot y = \varphi x + R_c \beta y = \varphi x + \beta' y$ , where  $\beta' = R_c \beta$ .

2. We can re-write the form  $x \cdot y = \alpha x + \psi y + c$  of a right linear quasigroup  $(Q, \cdot)$  as follows  $x \cdot y = \alpha x + c - c + \psi y + c = R_c \alpha x + I_c \psi y = \alpha' x + \psi' y$ , where  $I_c \psi y = -c + \psi y + c$ .

3. The proof is similar to the proof of Case 1.

4. The proof is similar to the proof of Case 2.

□

**Remark 14.** In general Lemma 13 is not true for linear, left linear right alinear, left alinear right linear, and alinear quasigroups.

## 2 Parastrophes of left(right) linear (alinear) quasigroups

**Lemma 15.** *Suppose that quasigroup  $(Q, \cdot)$  is linear with the form  $x \cdot y = \varphi x + \psi y + c$  over a group  $(Q, +)$ . Then its parastrophes have the following forms:*

1.  $x \stackrel{(12)}{\cdot} y = \varphi y + \psi x + c;$
2.  $x \stackrel{(13)}{\cdot} y = \varphi^{-1}x + IJ_{I\varphi^{-1}c}\varphi^{-1}\psi y + I\varphi^{-1}c$ , where  $J_{I\varphi^{-1}c}x = -\varphi^{-1}c + x + \varphi^{-1}c;$
3.  $x \stackrel{(23)}{\cdot} y = I\psi^{-1}\varphi x + \psi^{-1}y + I\psi^{-1}c;$
4.  $x \stackrel{(123)}{\cdot} y = \varphi^{-1}y + IJ_{I\varphi^{-1}c}\varphi^{-1}\psi x + I\varphi^{-1}c;$
5.  $x \stackrel{(132)}{\cdot} y = I\psi^{-1}\varphi y + \psi^{-1}x + I\psi^{-1}c.$

*Proof.* Case 1 is clear.

Case 2. By definition of parastrophy  $x \cdot y = z \Leftrightarrow z \stackrel{(13)}{\cdot} y = x$ . From equality  $x \cdot y = \varphi x + \psi y + c = z$  we have  $\varphi x = z - c - \psi y$ ,  $x = z \stackrel{(13)}{\cdot} y = \varphi^{-1}z - \varphi^{-1}c - \varphi^{-1}\psi y$ . If we replace the letter  $z$  by the letter  $x$ , we obtain  $x \stackrel{(13)}{\cdot} y = \varphi^{-1}x - \varphi^{-1}c - \varphi^{-1}\psi y = \varphi^{-1}x - \varphi^{-1}c - \varphi^{-1}\psi y + \varphi^{-1}c - \varphi^{-1}c = \varphi^{-1}x + J_{I\varphi^{-1}c}I\varphi^{-1}\psi y + I\varphi^{-1}c = \varphi^{-1}x + IJ_{I\varphi^{-1}c}\varphi^{-1}\psi y + I\varphi^{-1}c$ .

Case 3. By definition of parastrophy  $x \cdot y = z \Leftrightarrow x \stackrel{(23)}{\cdot} z = y$ . From equality  $x \cdot y = \varphi x + \psi y + c = z$  we have  $\psi y = -\varphi x + z - c$ ,  $y = x \stackrel{(23)}{\cdot} z = \psi^{-1}I\varphi x + \psi^{-1}z + \psi^{-1}Ic$ . If we replace the letter  $z$  by the letter  $y$ , we obtain  $x \stackrel{(23)}{\cdot} y = \psi^{-1}I\varphi x + \psi^{-1}y + \psi^{-1}Ic = I\psi^{-1}\varphi x + \psi^{-1}y + I\psi^{-1}c$ .

Cases 4 and 5 are "(12)-parastrophes" of Cases 2 and 3, respectively.  $\square$

Recall, by Lemma 12,  $\overline{\varphi} = I\varphi$ ,  $\overline{\psi} = I\psi$ . Below we shall use this relations without additional comments.

**Lemma 16.** *Suppose that quasigroup  $(Q, \cdot)$  is alinear with the form  $x \cdot y = \overline{\varphi}x + \overline{\psi}y + c$  over a group  $(Q, +)$ . Then its parastrophes have the following forms:*

1.  $x \stackrel{(12)}{\cdot} y = \overline{\varphi}y + \overline{\psi}x + c;$
2.  $x \stackrel{(13)}{\cdot} y = I\varphi^{-1}\psi y + IJ_{\varphi^{-1}c}\varphi^{-1}x + \varphi^{-1}c;$
3.  $x \stackrel{(23)}{\cdot} y = IJ_{\psi^{-1}c}\psi^{-1}y + IJ_{\psi^{-1}c}\psi^{-1}\varphi x + \psi^{-1}c;$
4.  $x \stackrel{(123)}{\cdot} y = I\varphi^{-1}\psi x + IJ_{\varphi^{-1}c}\varphi^{-1}y + \varphi^{-1}c;$
5.  $x \stackrel{(132)}{\cdot} y = IJ_{\psi^{-1}c}\psi^{-1}x + IJ_{\psi^{-1}c}\psi^{-1}\varphi y + \psi^{-1}c.$

*Proof.* Case 1 is clear.

Case 2. By definition of parastrophy  $x \cdot y = z \Leftrightarrow z \stackrel{(13)}{\cdot} y = x$ . From equality  $x \cdot y = \overline{\varphi}x + \overline{\psi}y + c = z$  we have  $\overline{\varphi}x = z - c - \overline{\psi}y$ ,  $x = z \stackrel{(13)}{\cdot} y = -I\varphi^{-1}\overline{\psi}y - I\varphi^{-1}c + I\varphi^{-1}z = \varphi^{-1}\overline{\psi}y + \varphi^{-1}c - \varphi^{-1}z = I\varphi^{-1}\psi y + \varphi^{-1}c + I\varphi^{-1}z = I\varphi^{-1}\psi y + J_{\varphi^{-1}c}I\varphi^{-1}z + \varphi^{-1}c$ .

If we replace the letter  $z$  by the letter  $x$ , we obtain  $x \stackrel{(13)}{\cdot} y = I\varphi^{-1}\psi y + IJ_{\varphi^{-1}c}\varphi^{-1}x + \varphi^{-1}c$ .

Case 3. By definition of parastrophy  $x \cdot y = z \Leftrightarrow x \stackrel{(23)}{\cdot} z = y$ . From equality  $x \cdot y = I\varphi x + I\psi y + c = z$  we have  $I\psi y = -I\varphi x + z - c$ ,  $y = x \stackrel{(23)}{\cdot} z = I\psi^{-1}(-I\varphi x + z - c) = \psi^{-1}c + I\psi^{-1}z + I\psi^{-1}\varphi x$ .

If we replace the letter  $z$  by the letter  $y$ , we obtain  $x \stackrel{(23)}{\cdot} y = \psi^{-1}c + I\psi^{-1}y + I\psi^{-1}\varphi x = J_{\psi^{-1}c}I\psi^{-1}y + J_{\psi^{-1}c}I\psi^{-1}\varphi x + \psi^{-1}c = IJ_{\psi^{-1}c}\psi^{-1}y + IJ_{\psi^{-1}c}\psi^{-1}\varphi x + \psi^{-1}c$ .

Cases 4 and 5 are "(12)-parastrophes" of Cases 2 and 3, respectively.  $\square$

**Remark 17.** From Lemma 16 it follows that any parastrophe of an alinear quasigroup is alinear.

**Lemma 18.** Suppose that  $(Q, \cdot)$  is left linear right alinear quasigroup with the form  $x \cdot y = \varphi x + I\psi y + c$  over a group  $(Q, +)$ . Then its parastrophes have the following forms:

1.  $x \stackrel{(12)}{\cdot} y = \varphi y + I\psi x + c$ ;
2.  $x \stackrel{(13)}{\cdot} y = \varphi^{-1}x + J_{I\varphi^{-1}c}\varphi^{-1}\psi y + I\varphi^{-1}c$ ;
3.  $x \stackrel{(23)}{\cdot} y = IJ_{\psi^{-1}c}\psi^{-1}y + J_{\psi^{-1}c}\psi^{-1}\varphi x + \psi^{-1}c$ ;
4.  $x \stackrel{(123)}{\cdot} y = \varphi^{-1}y + J_{I\varphi^{-1}c}\varphi^{-1}\psi x + I\varphi^{-1}c$ ;
5.  $x \stackrel{(132)}{\cdot} y = IJ_{\psi^{-1}c}\psi^{-1}x + J_{\psi^{-1}c}\psi^{-1}\varphi y + \psi^{-1}c$ .

*Proof.* Case 1 is clear.

Case 2. From equality  $x \cdot y = \varphi x + \overline{\psi}y + c = z$  we have  $\varphi x = z - c - \overline{\psi}y$ ,  $x = z \stackrel{(13)}{\cdot} y = \varphi^{-1}z - \varphi^{-1}c + \varphi^{-1}\psi y = \varphi^{-1}z + J_{I\varphi^{-1}c}\varphi^{-1}\psi y + I\varphi^{-1}c$ .

If we replace the letter  $z$  by the letter  $x$ , we obtain  $x \stackrel{(13)}{\cdot} y = \varphi^{-1}x + J_{I\varphi^{-1}c}\varphi^{-1}\psi y + I\varphi^{-1}c$ .

Case 3. From equality  $x \cdot y = \varphi x + I\psi y + c = z$  we have  $I\psi y = I\varphi x + z - c$ ,  $y = x \stackrel{(23)}{\cdot} z = IJ_{\psi^{-1}c}\psi^{-1}z + J_{\psi^{-1}c}\psi^{-1}\varphi x + \psi^{-1}c$ .

If we replace the letter  $z$  by the letter  $y$ , then we obtain  $x \stackrel{(23)}{\cdot} y = IJ_{\psi^{-1}c}\psi^{-1}y + J_{\psi^{-1}c}\psi^{-1}\varphi x + \psi^{-1}c$ .

Cases 4 and 5 are "(12)-parastrophes" of Cases 2 and 3, respectively.  $\square$

**Lemma 19.** Suppose that  $(Q, \cdot)$  is left alinear right linear quasigroup with the form  $x \cdot y = I\varphi x + \psi y + c$  over a group  $(Q, +)$ . Then its parastrophes have the following forms:

1.  $x \stackrel{(12)}{\cdot} y = I\varphi y + \psi x + c$ ;
2.  $x \stackrel{(13)}{\cdot} y = \varphi^{-1}\psi y + IJ_{\varphi^{-1}c}\varphi^{-1}x + \varphi^{-1}c$ ;
3.  $x \stackrel{(23)}{\cdot} y = \psi^{-1}\varphi x + \psi^{-1}y + I\psi^{-1}c$ ;
4.  $x \stackrel{(123)}{\cdot} y = \varphi^{-1}\psi x + IJ_{\varphi^{-1}c}\varphi^{-1}y + \varphi^{-1}c$ ;
5.  $x \stackrel{(132)}{\cdot} y = \psi^{-1}\varphi y + \psi^{-1}x + I\psi^{-1}c$ .

*Proof.* Case 1 is clear.

Case 2. We have  $x = z \stackrel{(13)}{\cdot} y = \varphi^{-1}\psi y + IJ_{\varphi^{-1}c}\varphi^{-1}z + \varphi^{-1}c$ . If we replace the letter  $z$  by the letter  $x$ , we obtain  $x \stackrel{(13)}{\cdot} y = \varphi^{-1}\psi y + IJ_{\varphi^{-1}c}\varphi^{-1}x + \varphi^{-1}c$ .

Case 3. From equality  $x \cdot y = I\varphi x + \psi y + c = z$  we have  $y = \psi^{-1}\varphi x + \psi^{-1}z + I\psi^{-1}c$ .

If we replace the letter  $z$  by the letter  $y$ , we obtain  $x \stackrel{(23)}{\cdot} y = \psi^{-1}\varphi x + \psi^{-1}y + I\psi^{-1}c$ .

Cases 4 and 5 are "(12)-parastrophes" of Cases 2 and 3, respectively.  $\square$

### 3 Orthogonality of left (right) linear (alinear) quasigroups

In this section we give conditions of orthogonality of a pair of left (right) linear quasigroups over a group  $(Q, +)$ .

**Definition 20.** Binary groupoid  $(G, \circ)$  is isotopic image of a binary groupoid  $(G, \cdot)$ , if there exist permutations  $\alpha, \beta, \gamma$  of the set  $Q$  such that  $x \circ y = \gamma^{-1}(\alpha x \cdot \beta y)$ . The ordered triple of permutations  $(\alpha, \beta, \gamma)$  of the set  $Q$  is called an *isotopy*.

In Lemma 21 a square is the inner part of Cayley table of a finite groupoid.

**Lemma 21.** Squares  $S_1(Q_1)$  and  $S_2(Q_2)$  are orthogonal if and only if their isotopic images are orthogonal with the isotopies of the form  $T_1 = (\varepsilon, \varepsilon, \varphi)$  and  $T_2 = (\varepsilon, \varepsilon, \psi)$ , respectively [9, Lemma 7].

**Lemma 22.** By the study of orthogonality of left (right) linear (alinear) quasigroups of quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  of the forms  $x \cdot y = \alpha x + \beta y + c$  and  $x \circ y = \gamma x + \delta y + d$  we can take  $c = d = 0$  without loss of generality.

*Proof.* The inner part of Cayley table of any quasigroup is a square in the sense of Lemma 21. Quasigroup  $(Q, \cdot)$  is isotope of the form  $(\varepsilon, \varepsilon, R_c^{-1})$  of quasigroup  $(Q, \diamond)$  with the form  $x \diamond y = \alpha x + \beta y$ .

Quasigroup  $(Q, \circ)$  is isotope of the form  $(\varepsilon, \varepsilon, R_d^{-1})$  of quasigroup  $(Q, *)$  with the form  $x * y = \gamma x + \delta y$ .

Therefore by the study of orthogonality of left (right) linear (alinear) quasigroups we can take  $c = d = 0$  without loss of generality.  $\square$

By Lemma 13 any left linear quasigroup  $(Q, \cdot)$  over a group  $(Q, +)$  has the form  $x \cdot y = \varphi x + \beta y$ , where  $\varphi \in \text{Aut}(Q, +)$ ,  $\beta \in S_Q$ .

**Theorem 23.** Left linear quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  of the form  $x \cdot y = \varphi x + \beta y$  and  $x \circ y = \psi x + \delta y$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $(-\varphi^{-1}\beta + \psi^{-1}\delta)$  is a permutation of the set  $Q$ .

*Proof.* We follow [15, Theorem 7]. Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \varphi x + \beta y = a \\ \psi x + \delta y = b \end{cases} \quad (1)$$

has a unique solution for any fixed elements  $a, b \in Q$ . We solve this system of equations in the usual way.

$$\begin{cases} x + \varphi^{-1}\beta y = \varphi^{-1}a \\ x + \psi^{-1}\delta y = \psi^{-1}b \end{cases} \iff \begin{cases} -\varphi^{-1}\beta y - x = -\varphi^{-1}a \\ x + \psi^{-1}\delta y = \psi^{-1}b \end{cases}$$

We perform the following transformation: (I row + II row  $\rightarrow$  I row) and obtain the system:

$$\begin{cases} -\varphi^{-1}\beta y + \psi^{-1}\delta y = -\varphi^{-1}a + \psi^{-1}b \\ x + \psi^{-1}\delta y = \psi^{-1}b \end{cases}$$

Write expression  $-\varphi^{-1}\beta y + \psi^{-1}\delta y$  as follows  $(-\varphi^{-1}\beta + \psi^{-1}\delta)y$ . Then the system (1) is equivalent to the following system

$$\begin{cases} (-\varphi^{-1}\beta + \psi^{-1}\delta)y = -\varphi^{-1}a + \psi^{-1}b \\ x + \psi^{-1}\delta y = \psi^{-1}b \end{cases}$$

It is clear that the system (1) has a unique solution if and only if the mapping  $(-\varphi^{-1}\beta + \psi^{-1}\delta)$  is a permutation of the set  $Q$ .  $\square$

**Theorem 24.** *Linear quasigroups  $(Q, \cdot)$  and left linear quasigroup  $(Q, \circ)$  of the form  $x \cdot y = \varphi x + \beta y + c$  and  $x \circ y = \psi y + \delta x$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $(J_{\psi^{-1}b}\varphi^{-1}\delta - \beta^{-1}\varphi)$  is a permutation of the set  $Q$  for any  $b \in Q$ .*

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \varphi x + \beta y = a - c \\ \psi y + \delta x = b \end{cases} \quad (2)$$

has a unique solution for any fixed elements  $a, b \in Q$ . We solve this system of equations in the usual way.

$$\begin{cases} Iy + I\beta^{-1}\varphi x = I\beta^{-1}(a - c) \\ y + \psi^{-1}\delta x = \psi^{-1}b \end{cases} \iff \begin{cases} Iy + I\beta^{-1}\varphi x = I\beta^{-1}(a - c) \\ J_y\psi^{-1}\delta x + y = \psi^{-1}b \end{cases}$$

We perform the following transformation: (II row + I row  $\rightarrow$  I row) and obtain the system:

$$\begin{cases} J_y\psi^{-1}\delta x + I\beta^{-1}\varphi x = \psi^{-1}b + I\beta^{-1}(a - c) \\ y = \psi^{-1}b - J_y\psi^{-1}\delta x \end{cases} \quad (3)$$

Substitute the right side of the second equation of the system (3) in the first equation of this system instead of the variable  $y$ :

Then the system (3) is equivalent to the following system

$$\begin{cases} \psi^{-1}b - \varphi^{-1}\delta x + \varphi^{-1}\delta x + \varphi^{-1}\delta x - \psi^{-1}b - \beta^{-1}\varphi x = \psi^{-1}b + I\beta^{-1}(a - c) \\ y = \psi^{-1}b - J_y\psi^{-1}\delta x \end{cases}$$

Further we have

$$\begin{cases} J_{\psi^{-1}b}\varphi^{-1}\delta x - \beta^{-1}\varphi x = \psi^{-1}b + I\beta^{-1}(a - c) \\ y = \psi^{-1}b - J_y\psi^{-1}\delta x \end{cases}$$

Write expression  $(J_{\psi^{-1}b}\varphi^{-1}\delta x - \beta^{-1}\varphi x)$  as follows  $(J_{\psi^{-1}b}\varphi^{-1}\delta - \beta^{-1}\varphi)x$  and remember that  $y + \psi^{-1}\delta x = \psi^{-1}b$ .

$$\begin{cases} (J_{\psi^{-1}b}\varphi^{-1}\delta - \beta^{-1}\varphi)x = \psi^{-1}b - \beta^{-1}(a - c) \\ y = \psi^{-1}b - \psi^{-1}\delta x \end{cases} \quad (4)$$

The systems (2) and (4) are equivalent. It is clear that the system (2) has a unique solution if and only if the mapping  $(J_{\psi^{-1}b}\varphi^{-1}\delta - \beta^{-1}\varphi)$  is a permutation of the set  $Q$  for any  $b \in Q$ .  $\square$



**Remark 25.** If in conditions of Theorem 24  $(Q, +)$  is an abelian group, then expression  $(J_{\psi^{-1}b}\varphi^{-1}\delta - \beta^{-1}\varphi)$  takes the form  $(\varphi^{-1}\delta - \beta^{-1}\varphi)$ , since in any abelian group any inner automorphism is the identity automorphism.

**Remark 26.** Even in the case, when the group  $(Q, +)$  is a finite cyclic group, the solution of equation  $\psi^{-1}\delta x - \beta^{-1}\varphi x = \psi^{-1}b - \beta^{-1}(a - c)$  is sufficiently complicate computational problem, since in general case permutation  $\delta$  is not an automorphism of the group  $(Q, +)$ , This remark also applies to the similar theorems that are given below.

By Lemma 13 any right linear quasigroup  $(Q, \cdot)$  over a group  $(Q, +)$  has the form  $x \cdot y = \alpha x + \varphi y$ , where  $\alpha \in S_Q$ ,  $\varphi \in \text{Aut}(Q, +)$ .

**Theorem 27.** Right linear quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  of the form  $x \cdot y = \alpha x + \varphi y$  and  $x \circ y = \gamma x + \psi y$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $(\varphi^{-1}\alpha - \psi^{-1}\gamma)$  is a permutation of the set  $Q$ .

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \alpha x + \varphi y = a \\ \gamma x + \psi y = b \end{cases} \quad (5)$$

has a unique solution for any fixed elements  $a, b \in Q$ . We solve this system of equations in the usual way.

$$\begin{cases} \varphi^{-1}\alpha x + y = \varphi^{-1}a \\ \psi^{-1}\gamma x + y = \psi^{-1}b \end{cases} \iff \begin{cases} \varphi^{-1}\alpha x + y = \varphi^{-1}a \\ -y - \psi^{-1}\gamma x = -\psi^{-1}b \end{cases}$$

We do the following transformation: (I row + II row  $\rightarrow$  I row) and obtain the system:

$$\begin{cases} \varphi^{-1}\alpha x - \psi^{-1}\gamma x = \varphi^{-1}a - \psi^{-1}b \\ -y - \psi^{-1}\gamma x = -\psi^{-1}b. \end{cases}$$

Write expression  $\varphi^{-1}\alpha x - \psi^{-1}\gamma x$  as follows  $(\varphi^{-1}\alpha - \psi^{-1}\gamma)x$ . Then the system (5) is equivalent to the following system

$$\begin{cases} (\varphi^{-1}\alpha - \psi^{-1}\gamma)x = \varphi^{-1}a - \psi^{-1}b \\ -y - \psi^{-1}\gamma x = -\psi^{-1}b. \end{cases}$$

It is clear that the system (5) has a unique solution if and only if the mapping  $(\varphi^{-1}\alpha - \psi^{-1}\gamma)$  is a permutation of the set  $Q$ .  $\square$

**Theorem 28.** Right alinear quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  of the form  $x \cdot y = \alpha x + \overline{\varphi}y$  and  $x \circ y = \gamma x + \overline{\psi}y$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $(-\overline{(\varphi)}^{-1}\alpha + \overline{(\psi)}^{-1}\gamma)$  is a permutation of the set  $Q$ .

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \alpha x + \overline{\varphi}y = a \\ \gamma x + \overline{\psi}y = b \end{cases} \quad (6)$$

has a unique solution for any fixed elements  $a, b \in Q$ . We solve this system of equations in the usual way.

$$\begin{cases} y + (\overline{\varphi})^{-1}\alpha x = (\overline{\varphi})^{-1}a \\ y + (\overline{\psi})^{-1}\gamma x = (\overline{\psi})^{-1}b \end{cases} \iff \begin{cases} I(\overline{\varphi})^{-1}\alpha x + Iy = I(\overline{\varphi})^{-1}a \\ y + (\overline{\psi})^{-1}\gamma x = (\overline{\psi})^{-1}b. \end{cases}$$

We do the following transformation: (I row + II row  $\rightarrow$  I row) and obtain the system:

$$\begin{cases} I(\overline{\varphi})^{-1}\alpha x + (\overline{\psi})^{-1}\gamma x = I(\overline{\varphi})^{-1}a + (\overline{\psi})^{-1}b \\ y + (\overline{\psi})^{-1}\gamma x = (\overline{\psi})^{-1}b. \end{cases}$$

Write expression  $I(\overline{\varphi})^{-1}\alpha x + (\overline{\psi})^{-1}\gamma x$  as follows  $(-(\overline{\varphi})^{-1}\alpha + (\overline{\psi})^{-1}\gamma)x$ . Then the system (6) is equivalent to the following system

$$\begin{cases} (-(\overline{\varphi})^{-1}\alpha + (\overline{\psi})^{-1}\gamma)x = -(\overline{\varphi})^{-1}a + (\overline{\psi})^{-1}b \\ y + (\overline{\psi})^{-1}\gamma x = (\overline{\psi})^{-1}b. \end{cases}$$

It is clear that the system (6) has a unique solution if and only if the mapping  $(-(\overline{\varphi})^{-1}\alpha + (\overline{\psi})^{-1}\gamma)$  is a permutation of the set  $Q$ .  $\square$

**Theorem 29.** *Left alinear quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  of the form  $x \cdot y = \overline{\varphi}x + \beta y$  and  $x \circ y = \overline{\psi}x + \delta y$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $((\overline{\varphi})^{-1}\beta - (\overline{\psi})^{-1}\delta)$  is a permutation of the set  $Q$ .*

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \overline{\varphi}x + \beta y = a \\ \overline{\psi}x + \delta y = b \end{cases} \quad (7)$$

has a unique solution for any fixed elements  $a, b \in Q$ . We solve this system of equations in the usual way:

$$\begin{cases} (\overline{\varphi})^{-1}\beta y + x = (\overline{\varphi})^{-1}a \\ (\overline{\psi})^{-1}\delta y + x = (\overline{\psi})^{-1}b \end{cases} \iff \begin{cases} (\overline{\varphi})^{-1}\beta y + x = (\overline{\varphi})^{-1}a \\ Ix + I(\overline{\psi})^{-1}\delta y = I(\overline{\psi})^{-1}b \end{cases}$$

We do the following transformation: (I row + II row  $\rightarrow$  II row) and obtain the system:

$$\begin{cases} (\overline{\varphi})^{-1}\beta y + x = (\overline{\varphi})^{-1}a \\ (\overline{\varphi})^{-1}\beta y + I(\overline{\psi})^{-1}\delta y = (\overline{\varphi})^{-1}a - (\overline{\psi})^{-1}b \end{cases}$$

Write expression  $(\overline{\varphi})^{-1}\beta y + I(\overline{\psi})^{-1}\delta y$  as follows  $((\overline{\varphi})^{-1}\beta - (\overline{\psi})^{-1}\delta)y$ . Then the system (7) is equivalent to the following system:

$$\begin{cases} (\overline{\varphi})^{-1}\beta y + x = (\overline{\varphi})^{-1}a \\ ((\overline{\varphi})^{-1}\beta - (\overline{\psi})^{-1}\delta)y = (\overline{\varphi})^{-1}a - (\overline{\psi})^{-1}b \end{cases}$$

It is clear that the system (7) has a unique solution if and only if the mapping  $((\overline{\varphi})^{-1}\beta - (\overline{\psi})^{-1}\delta)$  is a permutation of the set  $Q$ .  $\square$

**Remark 30.** *The mapping  $((\overline{\varphi})^{-1}\beta - (\overline{\psi})^{-1}\delta)$  from Theorem 29 it is possible to write also in the form  $((\overline{\varphi})^{-1}\beta - (\overline{\psi})^{-1}\delta) = I\varphi^{-1}\beta - I\psi^{-1}\delta = I(-\psi^{-1}\delta + \varphi^{-1}\beta)$ .*

**Theorem 31.** *Left linear quasigroup  $(Q, \cdot)$  and right alinear quasigroup  $(Q, \circ)$  of the form  $x \cdot y = \varphi x + \beta y$  and  $x \circ y = \gamma y + \overline{\psi}x$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $(\psi^{-1}\gamma + \varphi^{-1}\beta)$  is a permutation of the set  $Q$ .*

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \varphi x + \beta y = a \\ \gamma y + \bar{\psi} x = b \end{cases} \quad (8)$$

has a unique solution for any fixed elements  $a, b \in Q$ . We solve this system of equations in the usual way.

$$\begin{cases} x + \varphi^{-1}\beta y = \varphi^{-1}a \\ x + (\bar{\psi})^{-1}\gamma y = (\bar{\psi})^{-1}b \end{cases} \iff \begin{cases} I\varphi^{-1}\beta y + Ix = I\varphi^{-1}a \\ x + (\bar{\psi})^{-1}\gamma y = (\bar{\psi})^{-1}b. \end{cases}$$

We make the following transformation: (I row + II row  $\rightarrow$  I row) and obtain the system:

$$\begin{cases} I\varphi^{-1}\beta y + (\bar{\psi})^{-1}\gamma y = I\varphi^{-1}a + (\bar{\psi})^{-1}b \\ x + (\bar{\psi})^{-1}\gamma y = (\bar{\psi})^{-1}b. \end{cases}$$

Write expression  $I\varphi^{-1}\beta y + (\bar{\psi})^{-1}\gamma y$  as follows  $(-\varphi^{-1}\beta + (\bar{\psi})^{-1}\gamma)y$ . Then the system (8) is equivalent to the following system

$$\begin{cases} (-\varphi^{-1}\beta + (\bar{\psi})^{-1}\gamma)y = I\varphi^{-1}a + (\bar{\psi})^{-1}b \\ x + (\bar{\psi})^{-1}\gamma y = (\bar{\psi})^{-1}b. \end{cases}$$

It is clear that the system (8) has a unique solution if and only if the mapping  $-\varphi^{-1}\beta + (\bar{\psi})^{-1}\gamma$  is a permutation of the set  $Q$ . We simplify the last equality.

$$-\varphi^{-1}\beta + (\bar{\psi})^{-1}\gamma = I\varphi^{-1}\beta + I\psi^{-1}\gamma = I(\psi^{-1}\gamma + \varphi^{-1}\beta)$$

Therefore the system (8) has a unique solution if and only if the mapping  $(\psi^{-1}\gamma + \varphi^{-1}\beta)$  is a permutation of the set  $Q$ .  $\square$

**Theorem 32.** *Left linear quasigroup  $(Q, \cdot)$  and right alinear quasigroup  $(Q, \circ)$  of the form  $x \cdot y = \varphi y + \beta x$  and  $x \circ y = \gamma x + \bar{\psi} y$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $(\psi^{-1}\gamma + \varphi^{-1}\beta)$  is a permutation of the set  $Q$ .*

*Proof.* The proof is similar to the proof of Theorem 31 and we omit it.  $\square$

**Theorem 33.** *Left linear quasigroup  $(Q, \cdot)$  and left alinear quasigroup  $(Q, \circ)$  of the form  $x \cdot y = \varphi x + \beta y$  and  $x \circ y = I\psi x + \delta y$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $(\psi^{-1}\delta + J_{I\psi^{-1}b}\varphi^{-1}\beta)$  is a permutation of the set  $Q$  for any  $b \in Q$ .*

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \varphi x + \beta y = a \\ \bar{\psi} x + \delta y = b \end{cases} \quad (9)$$

has a unique solution for any fixed elements  $a, b \in Q$ . We solve this system of equations in the usual way.

$$\begin{cases} x + \varphi^{-1}\beta y = \varphi^{-1}a \\ (\bar{\psi})^{-1}\delta y + x = (\bar{\psi})^{-1}b \end{cases} \iff \begin{cases} J_x\varphi^{-1}\beta y + x = \varphi^{-1}a \\ Ix + \psi^{-1}\delta y = \psi^{-1}b \end{cases}$$

We make the following transformation: (I row + II row  $\rightarrow$  I row) and obtain the system:

$$\begin{cases} J_x \varphi^{-1} \beta y + \psi^{-1} \delta y = \varphi^{-1} a + \psi^{-1} b \\ Ix = \psi^{-1} b + I\psi^{-1} \delta y \end{cases} \quad (10)$$

We simplify the left part of the first equation of system (10) using the second equation of this system:

$$\begin{aligned} J_x \varphi^{-1} \beta y + \psi^{-1} \delta y &= \\ x + \varphi^{-1} \beta y - x + \psi^{-1} \delta y &= \\ \psi^{-1} \delta y - \psi^{-1} b + \varphi^{-1} \beta y + \psi^{-1} b + I\psi^{-1} \delta y + \psi^{-1} \delta y &= \\ \psi^{-1} \delta y - \psi^{-1} b + \varphi^{-1} \beta y + \psi^{-1} b &= \\ \psi^{-1} \delta y + J_{I\psi^{-1}b} \varphi^{-1} \beta y. \end{aligned}$$

Write expression  $(\psi^{-1} \delta y + J_{I\psi^{-1}b} \varphi^{-1} \beta y)$  as follows  $(\psi^{-1} \delta + J_{I\psi^{-1}b} \varphi^{-1} \beta)y$ . Then the system (10) is equivalent to the following system

$$\begin{cases} (\psi^{-1} \delta + J_{I\psi^{-1}b} \varphi^{-1} \beta)y = \varphi^{-1} a + \psi^{-1} b \\ Ix = \psi^{-1} b + I\psi^{-1} \delta y. \end{cases}$$

It is clear that the system (9) has a unique solution if and only if the mapping  $(\psi^{-1} \delta + J_{I\psi^{-1}b} \varphi^{-1} \beta)$  is a permutation of the set  $Q$  for any  $b \in Q$ .  $\square$

**Theorem 34.** *Left alinear quasigroup  $(Q, \cdot)$  and right linear quasigroup  $(Q, \circ)$  of the form  $x \cdot y = \overline{\varphi}x + \beta y$  and  $x \circ y = \gamma y + \psi x$ , respectively, which are defined over a group  $(Q, +)$ , are orthogonal if and only if the mapping  $(\psi^{-1} \gamma + \varphi^{-1} \beta)$  is a permutation of the set  $Q$ .*

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \overline{\varphi}x + \beta y = a \\ \gamma y + \psi x = b \end{cases} \quad (11)$$

has a unique solution for any fixed elements  $a, b \in Q$ . We solve this system of equations in the usual way:

$$\begin{cases} (\overline{\varphi})^{-1} \beta y + x = (\overline{\varphi})^{-1} a \\ \psi^{-1} \gamma y + x = \psi^{-1} b \end{cases} \iff \begin{cases} (\overline{\varphi})^{-1} \beta y + x = (\overline{\varphi})^{-1} a \\ -x - \psi^{-1} \gamma y = -\psi^{-1} b. \end{cases}$$

We do the following transformation: (I row + II row  $\rightarrow$  I row) and obtain the system:

$$\begin{cases} (\overline{\varphi})^{-1} \beta y - \psi^{-1} \gamma y = (\overline{\varphi})^{-1} a - \psi^{-1} b \\ -x - \psi^{-1} \gamma y = -\psi^{-1} b. \end{cases}$$

Write expression  $(\overline{\varphi})^{-1} \beta y - \psi^{-1} \gamma y$  as follows  $((\overline{\varphi})^{-1} \beta - \psi^{-1} \gamma)y$ . Then the system (11) is equivalent to the following system

$$\begin{cases} ((\overline{\varphi})^{-1} \beta - \psi^{-1} \gamma)y = (\overline{\varphi})^{-1} a - \psi^{-1} b \\ -x - \psi^{-1} \gamma y = -\psi^{-1} b. \end{cases}$$

It is clear that the system (11) has a unique solution if and only if the mapping  $((\overline{\varphi})^{-1} \beta - \psi^{-1} \gamma) = (I\varphi^{-1} \beta + I\psi^{-1} \gamma) = I(\psi^{-1} \gamma + \varphi^{-1} \beta)$  is a permutation of the set  $Q$ .  $\square$

Theorem 7 from [15] on conditions of orthogonality of linear quasigroups follows from Theorems 27 and 23.

**Theorem 35.** A linear quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \alpha x + \beta y + c$  and a linear quasigroup  $(Q, \circ)$  of the form  $x \circ y = \gamma y + \delta x + d$ , both defined over a group  $(Q, +)$ , are orthogonal if and only if the map  $(-J_t \gamma^{-1} \delta + \beta^{-1} \alpha)$  is a permutation of the set  $Q$  for any element  $t \in Q$ .

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \alpha x + \beta y + c = a \\ \gamma y + \delta x + d = b \end{cases}$$

has a unique solution for any fixed elements  $a, b \in Q$ .

We solve this system of equations as follows:

$$\begin{cases} \alpha x + \beta y = a - c \\ \gamma y + \delta x = b - d \end{cases} \iff \begin{cases} \beta^{-1} \alpha x + y = \beta^{-1}(a - c) \\ J_{\gamma y} \delta x + \gamma y = (b - d), \end{cases}$$

where  $J_{\gamma y} \delta x = \gamma y + \delta x - \gamma y$ . Notice  $\gamma^{-1} J_{\gamma y} \delta x = J_y \gamma^{-1} \delta x$ .

Further we have:

$$\begin{cases} \beta^{-1} \alpha x + y = \beta^{-1}(a - c) \\ -y - \gamma^{-1} J_{\gamma y} \delta x = -\gamma^{-1}(b - d). \end{cases} \quad (12)$$

If in the system (12) we add the first and the second row and write the sum instead of second row ( $I + II \rightarrow II$ ), then we obtain the following system

$$\begin{cases} \beta^{-1} \alpha x + y = \beta^{-1}(a - c) \\ \beta^{-1} \alpha x - \gamma^{-1} J_{\gamma y} \delta x = \beta^{-1}(a - c) - \gamma^{-1}(b - d). \end{cases} \quad (13)$$

Therefore we can rewrite the system (13) in the following form

$$\begin{cases} y = -\beta^{-1} \alpha x + \beta^{-1}(a - c) \\ \beta^{-1} \alpha x - J_y \gamma^{-1} \delta x = \beta^{-1}(a - c) - \gamma^{-1}(b - d). \end{cases} \quad (14)$$

Rewrite the left part of the second equation of the system (14) in the following form

$$\beta^{-1} \alpha x + I J_y \gamma^{-1} \delta x = \beta^{-1} \alpha x + J_y I \gamma^{-1} \delta x = \beta^{-1} \alpha x + y - \gamma^{-1} \delta x - y.$$

Further, taking into consideration first equation of the system (14), we have:

$$\begin{aligned} & \beta^{-1} \alpha x + y - \gamma^{-1} \delta x - y = \\ & \beta^{-1} \alpha x - \beta^{-1} \alpha x + \beta^{-1}(a - c) - \gamma^{-1} \delta x - \beta^{-1}(a - c) + \beta^{-1} \alpha x = \\ & \beta^{-1}(a - c) - \gamma^{-1} \delta x - \beta^{-1}(a - c) + \beta^{-1} \alpha x = \\ & J_{\beta^{-1}(a-c)} I \gamma^{-1} \delta x + \beta^{-1} \alpha x = -J_{\beta^{-1}(a-c)} \gamma^{-1} \delta x + \beta^{-1} \alpha x. \end{aligned}$$

Similarly, as in Theorem 23, we write expression  $-J_{\beta^{-1}(a-c)} \gamma^{-1} \delta x + \beta^{-1} \alpha x$  in the following form  $(-J_{\beta^{-1}(a-c)} \gamma^{-1} \delta + \beta^{-1} \alpha)x$ . The system (14) takes the form

$$\begin{cases} y = -\beta^{-1} \alpha x + \beta^{-1}(a - c) \\ (-J_{\beta^{-1}(a-c)} \gamma^{-1} \delta + \beta^{-1} \alpha)x = \beta^{-1}(a - c) - \gamma^{-1}(b - d). \end{cases} \quad (15)$$

From the system (15) it follows that quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the map  $(-J_{\beta^{-1}(a-c)} \gamma^{-1} \delta + \beta^{-1} \alpha)$  is a permutation of the set  $Q$  for any element  $a \in Q$ .

Denote the expression  $\beta^{-1}(a - c)$  by the letter  $t$ . We can reformulate the last condition as follows: quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the map  $(-J_t \gamma^{-1} \delta + \beta^{-1} \alpha)$  is a permutation of the set  $Q$  for any element  $t \in Q$ .  $\square$

Taking into consideration that we have not proved an analogue of Lemma 13 for linear quasigroup, we give independent from Theorems 27 and 23 proof of the following

**Theorem 36.** [15]. *A linear quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \alpha x + \beta y + c$  and a linear quasigroup  $(Q, \circ)$  of the form  $x \circ y = \gamma x + \delta y + d$ , both defined over a group  $(Q, +)$ , are orthogonal if and only if the map  $(-\gamma^{-1}\delta + \alpha^{-1}\beta)$  is a permutation of the set  $Q$ .*

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \alpha x + \beta y + c = a \\ \gamma x + \delta y + d = b \end{cases}$$

has a unique solution for any fixed elements  $a, b \in Q$ .

We solve this system of equations as follows:

$$\begin{cases} \alpha x + \beta y = a - c \\ \gamma x + \delta y = b - d \end{cases} \iff \begin{cases} x + \alpha^{-1}\beta y = \alpha^{-1}(a - c) \\ -\gamma^{-1}\delta y - x = -\gamma^{-1}(b - d). \end{cases}$$

In the last system we add the second and the first row and write the sum instead of second row ( $II + I \rightarrow II$ ). We obtain the following system

$$\begin{cases} x + \alpha^{-1}\beta y = \alpha^{-1}(a - c) \\ -\gamma^{-1}\delta y + \alpha^{-1}\beta y = -\gamma^{-1}(b - d) + \alpha^{-1}(a - c). \end{cases} \quad (16)$$

Similarly as in Theorem 23 we write expression  $-\gamma^{-1}\delta y + \alpha^{-1}\beta y$  in the following form  $(-\gamma^{-1}\delta + \alpha^{-1}\beta)y$ . From the system (16) it follows that quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the map  $(-\gamma^{-1}\delta + \alpha^{-1}\beta)$  is a permutation of the set  $Q$ .  $\square$

**Theorem 37.** *A  $T$ -quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \alpha x + \beta y + c$  and a  $T$ -quasigroup  $(Q, \circ)$  of the form  $x \circ y = \gamma x + \delta y + d$ , both defined over a group  $(Q, +)$ , are orthogonal if and only if the map  $\alpha^{-1}\beta - \gamma^{-1}\delta$  is an automorphism of the group  $(Q, +)$  [8, Theorem 16].*

*Proof.* The proof follows from Theorem 36 and the fact that in abelian group  $-\gamma^{-1}\delta + \alpha^{-1}\beta = \alpha^{-1}\beta - \gamma^{-1}\delta$  and that the map  $\alpha^{-1}\beta - \gamma^{-1}\delta$  is an endomorphism of the group  $(Q, +)$ .  $\square$

**Lemma 38.** *If  $(Q, +)$  is an abelian group,  $\varphi, \psi \in \text{Aut}(Q, +)$ , then  $\varphi - \psi$  is an automorphism of the group  $(Q, +)$  if and only if  $\psi - \varphi$  is an automorphism of this group.*

*Proof.* Taking into consideration that the map  $I(x) = -x$  is an automorphism of an abelian group  $(Q, +)$  and a permutation of the set  $Q$  of order two, we have  $-(\varphi - \psi) = -\varphi + \psi = \psi - \varphi$ .  $\square$

**Lemma 39.** *In conditions of Theorem 37 the following statements are equivalent: "the endomorphism  $(\alpha^{-1}\beta - \gamma^{-1}\delta)$  is an automorphism of  $(Q, +)$ " and "the endomorphism  $(\beta^{-1}\alpha - \delta^{-1}\gamma)$  is an automorphism of  $(Q, +)$ ".*

*Proof.* The map  $(\alpha^{-1}\beta - \gamma^{-1}\delta)$  is a permutation of the set  $Q$  if and only if the map  $\varepsilon - \alpha\gamma^{-1}\delta\beta^{-1}$  is a permutation of the set  $Q$ . Indeed,  $\alpha(\alpha^{-1}\beta - \gamma^{-1}\delta)\beta = \varepsilon - \alpha\gamma^{-1}\delta\beta^{-1}$ .

Similarly,  $(\beta^{-1}\alpha - \delta^{-1}\gamma)$  is a permutation of set  $Q$  if and only if the map

$$\varepsilon - \beta\delta^{-1}\gamma\alpha^{-1} \quad (17)$$

is a permutation of the set  $Q$ .

If we denote the map  $\alpha\gamma^{-1}\delta\beta^{-1}$  by  $\psi$ , then  $\beta\delta^{-1}\gamma\alpha^{-1} = \psi^{-1}$ .

Further we have the following equivalence: the map  $\varepsilon - \psi$  is a permutation if and only if the map  $\varepsilon - \psi^{-1}$  is a permutation of the set  $Q$ .

Indeed,  $\varepsilon - \psi$  is a permutation if and only if the map  $\psi - \varepsilon$  is a permutation (Lemma 38), further  $\psi - \varepsilon$  is a permutation if and only if  $\psi^{-1}(\psi - \varepsilon) = \varepsilon - \psi^{-1}$  is a permutation.  $\square$

**Corollary 40.** *A  $T$ -quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \varphi x + \psi y + c$  over a group  $(Q, +)$  and its (12)-parastrophe  $(Q, \star)$  of the form  $x \star y = \psi x + \varphi y + c$  are orthogonal if and only if the map  $\varphi^{-1}\psi - \psi^{-1}\varphi$  is an automorphism of the group  $(Q, +)$ .*

**Corollary 41.** *A  $T$ -quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \alpha x + \beta y + c$  and a medial quasigroup  $(Q, \circ)$  of the form  $x \circ y = \gamma x + \delta y + d$ , both over a group  $(Q, +)$ , are orthogonal if and only if the map  $\alpha\delta - \gamma\beta$  is an automorphism of the group  $(Q, +)$ .*

*Proof.* From equality (17) it follows that quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the map  $\varepsilon - \beta\delta^{-1}\gamma\alpha^{-1}$  is a permutation of the set  $Q$ . Further, since  $\delta\gamma = \gamma\delta$ , we have  $\beta\delta^{-1}\gamma\alpha^{-1} = \beta\gamma\delta^{-1}\alpha^{-1}$  and the map  $\varepsilon - \beta\delta^{-1}\gamma\alpha^{-1}$  is a permutation of the set  $Q$  if and only if the map  $(\varepsilon - \beta\gamma\delta^{-1}\alpha^{-1})\alpha\delta = \alpha\delta - \beta\gamma$  is a permutation of the set  $Q$ .  $\square$

**Theorem 42.** *A linear quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = I\alpha x + I\beta y + c$  and a linear quasigroup  $(Q, \circ)$  of the form  $x \circ y = I\gamma y + I\delta x + d$ , both defined over a group  $(Q, +)$ , where  $\alpha, \beta, \gamma, \delta \in \text{Aut}(Q, +)$ , are orthogonal if and only if the map  $(\beta^{-1}\alpha - J_t\gamma^{-1}\delta)$  is a permutation of the set  $Q$  for any element  $t \in Q$ .*

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} I\alpha x + I\beta y + c = a \\ I\gamma y + I\delta x + d = b \end{cases}$$

has a unique solution for any fixed elements  $a, b \in Q$ .

We solve this system of equations as follows:

$$\begin{cases} I\alpha x + I\beta y = a - c \\ I\gamma y + I\delta x = b - d \end{cases} \iff \begin{cases} y + \beta^{-1}\alpha x = I\beta^{-1}(a - c) \\ J_{I\gamma y}I\delta x + I\gamma y = (b - d), \end{cases}$$

where  $J_{I\gamma y}I\delta x = I\gamma y + I\delta x - I\gamma y$ . Notice  $\gamma^{-1}J_{I\gamma y}I\delta x = J_{Iy}\gamma^{-1}I\delta x$ .

Further we have:

$$\begin{cases} y + \beta^{-1}\alpha x = I\beta^{-1}(a - c) \\ J_{Iy}\gamma^{-1}I\delta x + Iy = \gamma^{-1}(b - d). \end{cases} \quad (18)$$

If in the system (18) we add the second and the first row and write the sum instead of the second row ( $II + I \rightarrow II$ ), then we obtain the following system

$$\begin{cases} y + \beta^{-1}\alpha x = I\beta^{-1}(a - c) \\ J_{Iy}\gamma^{-1}I\delta x + \beta^{-1}\alpha x = \gamma^{-1}(b - d) - \beta^{-1}(a - c). \end{cases} \quad (19)$$

Therefore we can rewrite the system (19) in the following form

$$\begin{cases} y = I\beta^{-1}(a - c) + I\beta^{-1}\alpha x \\ J_{Iy}\gamma^{-1}I\delta x + \beta^{-1}\alpha x = \gamma^{-1}(b - d) - \beta^{-1}(a - c). \end{cases} \quad (20)$$

Rewrite the left part of the second equation of the system (20) in the following form

$$\begin{aligned}
J_{Iy}\gamma^{-1}I\delta x + \beta^{-1}\alpha x &= -(J_{Iy}\gamma^{-1}\delta x) + \beta^{-1}\alpha x = \\
&= -(-y + \gamma^{-1}\delta x + y) + \beta^{-1}\alpha x = \\
&= -y - \gamma^{-1}\delta x + y + \beta^{-1}\alpha x \stackrel{(20)}{=} \\
&= -(I\beta^{-1}(a-c) + I\beta^{-1}\alpha x) - \gamma^{-1}\delta x + I\beta^{-1}(a-c) + I\beta^{-1}\alpha x + \beta^{-1}\alpha x = \\
&= \beta^{-1}\alpha x + \beta^{-1}(a-c) - \gamma^{-1}\delta x - \beta^{-1}(a-c) = \\
&= \beta^{-1}\alpha x - J_{\beta^{-1}(a-c)}\gamma^{-1}\delta x.
\end{aligned}$$

Similarly, as in Theorem 23, we write expression  $\beta^{-1}\alpha x - J_{\beta^{-1}(a-c)}\gamma^{-1}\delta x$  in the following form  $(\beta^{-1}\alpha - J_{\beta^{-1}(a-c)}\gamma^{-1}\delta)x$ . The system (20) takes the form

$$\begin{cases} y = I\beta^{-1}(a-c) + I\beta^{-1}\alpha x \\ (\beta^{-1}\alpha - J_{\beta^{-1}(a-c)}\gamma^{-1}\delta)x = \gamma^{-1}(b-d) - \beta^{-1}(a-c). \end{cases} \quad (21)$$

From system (21) it follows that quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the map  $(\beta^{-1}\alpha - J_{\beta^{-1}(a-c)}\gamma^{-1}\delta)$  is a permutation of the set  $Q$  for any element  $a \in Q$ .

Denote the expression  $\beta^{-1}(a-c)$  by the letter  $t$ . We can reformulate the last condition as follows: quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the map  $(\beta^{-1}\alpha - J_t\gamma^{-1}\delta)$  is a permutation of the set  $Q$  for any element  $t \in Q$ .  $\square$

**Theorem 43.** *Left linear quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \varphi x + \beta y$  and right linear quasigroup  $(Q, \circ)$  of the form  $x \circ y = \gamma y + \psi x$ , both defined over a group  $(Q, +)$ , where  $\varphi, \psi \in \text{Aut}(Q, +)$ , are orthogonal if and only if the map  $(J_t\psi^{-1}\gamma - \varphi^{-1}\beta)$  is a permutation of the set  $Q$  for any element  $t \in Q$ .*

*Proof.* Quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the system of equations

$$\begin{cases} \varphi x + \beta y = a \\ \gamma y + \psi x = b \end{cases}$$

has a unique solution for any fixed elements  $a, b \in Q$ .

We solve this system of equations as follows:

$$\begin{cases} I\varphi^{-1}\beta y + Ix = I\varphi^{-1}a \\ \psi^{-1}\gamma y + x = \psi^{-1}b \end{cases} \iff \begin{cases} I\varphi^{-1}\beta y + Ix = I\varphi^{-1}a \\ x + J_{-x}\psi^{-1}\gamma y = \psi^{-1}b \end{cases}$$

where  $J_{-x}\gamma y = -x + \gamma y + x$ .

If in the last system we add the first and the second equation and write the sum instead of the second equation  $(I + II \rightarrow II)$ , then we obtain the following system

$$\begin{cases} I\varphi^{-1}\beta y + Ix = I\varphi^{-1}a \\ I\varphi^{-1}\beta y + J_{-x}\psi^{-1}\gamma y = I\varphi^{-1}a + \psi^{-1}b. \end{cases} \quad (22)$$

Therefore we can rewrite system (22) in the following form

$$\begin{cases} x = \varphi^{-1}a + I\varphi^{-1}\beta y \\ I\varphi^{-1}\beta y + J_{-x}\psi^{-1}\gamma y = I\varphi^{-1}a + \psi^{-1}b. \end{cases} \quad (23)$$



Rewrite the left part of the second equation of system (23) in the following form

$$\begin{aligned}
& I\varphi^{-1}\beta y + J_{-x}\psi^{-1}\gamma y = \\
& I\varphi^{-1}\beta y - x + \psi^{-1}\gamma y + x = \\
& I\varphi^{-1}\beta y + \varphi^{-1}\beta y - \varphi^{-1}a + \psi^{-1}\gamma y + \varphi^{-1}a + I\varphi^{-1}\beta y = \\
& -\varphi^{-1}a + \psi^{-1}\gamma y + \varphi^{-1}a + I\varphi^{-1}\beta y = \\
& J_{I\varphi^{-1}a}\psi^{-1}\gamma y - \varphi^{-1}\beta y.
\end{aligned}$$

We write expression  $J_{I\varphi^{-1}a}\psi^{-1}\gamma y - \varphi^{-1}\beta y$  in the following form  $(J_{I\varphi^{-1}a}\psi^{-1}\gamma - \varphi^{-1}\beta)y$ . The system (23) takes the form

$$\begin{cases} x = \varphi^{-1}a + I\varphi^{-1}\beta y \\ (J_{I\varphi^{-1}a}\psi^{-1}\gamma - \varphi^{-1}\beta)y = I\varphi^{-1}a + \psi^{-1}b. \end{cases} \quad (24)$$

From system (24) it follows that quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the map  $(J_{I\varphi^{-1}a}\psi^{-1}\gamma - \varphi^{-1}\beta)$  is a permutation of the set  $Q$  for any element  $a \in Q$ .

Denote expression  $I\varphi^{-1}a$  by the letter  $t$ . We can reformulate the last condition as follows: quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$  are orthogonal if and only if the map  $(J_t\psi^{-1}\gamma - \varphi^{-1}\beta)$  is a permutation of the set  $Q$  for any element  $t \in Q$ .  $\square$

**Theorem 44.** *Left linear quasigroup  $(Q, \cdot)$  of the form  $x \cdot y = \varphi y + \beta x$  and right linear quasigroup  $(Q, \circ)$  of the form  $x \circ y = \gamma x + \psi y$ , both defined over a group  $(Q, +)$ , where  $\varphi, \psi \in \text{Aut}(Q, +)$ , are orthogonal if and only if the map  $(I\psi^{-1}\gamma + J_{I\psi^{-1}b}\varphi^{-1}\beta)$  is a permutation of the set  $Q$  for any element  $b \in Q$ .*

*Proof.* The proof is similar to the proof of Theorem 43 and we omit it.  $\square$

**Corollary 45.** *If in conditions of Theorem 35 (Theorem 42) the group  $\text{Inn}(Q, +)$  of inner automorphisms of the group  $(Q, +)$  acts on the group  $\text{Aut}(Q, +)$  transitively, then does not exist orthogonal quasigroups  $(Q, \cdot)$  and  $(Q, \circ)$ .*

*Proof.* Since the group  $\text{Inn}(Q, +)$  acts transitively, then in conditions of Theorem 35 there exists an element  $s \in Q$  such that  $J_s\gamma^{-1}\delta = \beta^{-1}\alpha$ , i.e., such that  $(-J_s\gamma^{-1}\delta + \beta^{-1}\alpha)x = 0$  for any  $x \in Q$ .

In conditions of Theorem 42 there exists an element  $d \in Q$  such that  $(\beta^{-1}\alpha - J_d\gamma^{-1}\delta)x = 0$  for any  $x \in Q$ .  $\square$

- Lemma 46.** 1. *If in conditions of Theorem 35 the group  $(Q, +)$  is symmetric group  $S_n$  ( $n \neq 2; 6$ ), then does not exist orthogonal quasigroups  $(S_n, \cdot)$  and  $(S_n, \circ)$ .*
2. *If in conditions of Theorem 42 the group  $(Q, +)$  is symmetric group  $S_n$  ( $n \neq 2; 6$ ), then does not exist orthogonal quasigroups  $(S_n, \cdot)$  and  $(S_n, \circ)$ .*

*Proof.* By Gölder theorem  $\text{Aut}(S_n) = \text{Inn}(S_n)$  for any natural number  $n$ ,  $n \neq 2; 6$  [p. 67][6].  $\square$

## 4 Orthogonality of parastrophes of left(right) linear(alinear) quasigroups

**Theorem 47.** *For a linear quasigroup  $(Q, A)$  of the form  $A(x, y) = \varphi x + \psi y + c$  over a group  $(Q, +)$  the following equivalences are fulfilled:*

1.  $A \perp A^{12} \iff$  the map  $(-J_t \varphi^{-1} \psi + \psi^{-1} \varphi)$  is a permutation of the set  $Q$  for any  $t \in Q$ ;
2.  $A \perp A^{13} \iff$  the map  $(\varphi J_{I\varphi^{-1}c} + \varepsilon)$  is a permutation of the set  $Q$ ;
3.  $A \perp A^{23} \iff$  the map  $(\varepsilon + \psi)$  is a permutation of the set  $Q$ ;
4.  $A \perp A^{123} \iff$  the map  $(\varphi J_{\psi^{-1}c}^{-1} + \psi^2)$  is a permutation of the set  $Q$ ;
5.  $A \perp A^{132} \iff$  the map  $(\varphi^2 + \psi)$  is a permutation of the set  $Q$ .

*Proof.* The forms of parastrophes of quasigroup  $(Q, A)$  are given in Lemma 15.

Case 1. The proof follows from Theorem 35.

Case 2. Using Theorem 23 we have:  $A \perp A^{13}$  if and only if the map  $I\varphi^{-1}\psi + \varphi IJ_{I\varphi^{-1}c}\varphi^{-1}\psi$  is a permutation of the set  $Q$ .

We make the following transformations:  $I\varphi^{-1}\psi + \varphi IJ_{I\varphi^{-1}c}\varphi^{-1}\psi = (I + \varphi IJ_{I\varphi^{-1}c})\varphi^{-1}\psi = (I + \varphi J_{I\varphi^{-1}c}I)\varphi^{-1}\psi = (\varphi J_{I\varphi^{-1}c} + \varepsilon)I\varphi^{-1}\psi$ . The last map is a permutation if and only if the map  $(\varphi J_{I\varphi^{-1}c} + \varepsilon)$  is a permutation of the set  $Q$ .

Case 3. Using Theorem 27 we have:  $A \perp A^{23}$  if and only if the map  $\psi^{-1}\varphi - \psi I\psi^{-1}\varphi$  is a permutation of the set  $Q$ . We simplify the last equality in the following way:

$$\psi^{-1}\varphi - \psi I\psi^{-1}\varphi = \psi^{-1}\varphi + \psi\psi^{-1}\varphi = (\varepsilon + \psi)\psi^{-1}\varphi.$$

Therefore  $A \perp A^{23}$  if and only if the map  $(\varepsilon + \psi)$  is a permutation of the set  $Q$ .

Case 4. From Theorem 31 it follows that  $A \perp A^{123}$  if and only if the map

$$I\varphi^{-1}\psi + (IJ_{\varphi^{-1}c}\varphi^{-1}\psi)^{-1}\varphi^{-1} \quad (25)$$

is a permutation of the set  $Q$ . We make the following transformation of expression (25):

$$\begin{aligned} I\varphi^{-1}\psi + (IJ_{\varphi^{-1}c}\varphi^{-1}\psi)^{-1}\varphi^{-1} &= \\ I\varphi^{-1}\psi + \psi^{-1}\varphi J_{\varphi^{-1}c}^{-1}I\varphi^{-1} &= \\ I(\psi^{-1}\varphi J_{\varphi^{-1}c}^{-1}\varphi^{-1} + \varphi^{-1}\psi) &= \\ I(\psi^{-1}J_c^{-1}\varphi\varphi^{-1} + \varphi^{-1}\psi) &= \\ I(\psi^{-1}J_c^{-1} + \varphi^{-1}\psi) &= \\ I\varphi^{-1}(\varphi\psi^{-1}J_c^{-1} + \psi) &= \\ I\varphi^{-1}(\varphi J_{\psi^{-1}c}^{-1}\psi^{-1} + \psi) &= \\ I\varphi^{-1}(\varphi J_{\psi^{-1}c}^{-1} + \psi^2)\psi^{-1}. \end{aligned} \quad (26)$$

We obtain:  $A \perp A^{123}$  if and only if the map  $(\varphi J_{\psi^{-1}c}^{-1} + \psi^2)$  is a permutation of the set  $Q$ .

Case 5. From Theorem 34 we have:  $A \perp A^{132}$  if and only if the map  $(I\psi^{-1}\varphi)^{-1}\psi^{-1} - \psi^{-1}\varphi$  is a permutation of the set  $Q$ . We simplify the last equality in the following way:

$$\begin{aligned} (I\psi^{-1}\varphi)^{-1}\psi^{-1} - \psi^{-1}\varphi &= \\ \varphi^{-1}\psi I\psi^{-1} + I\psi^{-1}\varphi &= \\ I\varphi^{-1} + I\psi^{-1}\varphi &= \\ I(\psi^{-1}\varphi + \varphi^{-1}) &= \\ I\psi^{-1}(\varphi^2 + \psi)\varphi^{-1}. \end{aligned}$$

Therefore  $A \perp A^{132}$  if and only if the map  $(\varphi^2 + \psi)$  is a permutation of the set  $Q$ . □

Taking into consideration Lemma 22 we can take in formulation of Theorem 47  $c = 0$  without loss of generality. Therefore we can reformulate Theorem 47 in the following form:

**Theorem 48.** *For a linear quasigroup  $(Q, A)$  of the form  $A(x, y) = \varphi x + \psi y + c$  over a group  $(Q, +)$  the following equivalences are fulfilled:*

1.  $A \perp A^{12} \iff$  the map  $(-J_t \varphi^{-1} \psi + \psi^{-1} \varphi)$  is a permutation of the set  $Q$  for any  $t \in Q$ ;
2.  $A \perp A^{13} \iff$  the map  $(\varphi + \varepsilon)$  is a permutation of the set  $Q$ ;
3.  $A \perp A^{23} \iff$  the map  $(\varepsilon + \psi)$  is a permutation of the set  $Q$ ;
4.  $A \perp A^{123} \iff$  the map  $(\varphi + \psi^2)$  is a permutation of the set  $Q$ ;
5.  $A \perp A^{132} \iff$  the map  $(\varphi^2 + \psi)$  is a permutation of the set  $Q$ .

**Corollary 49.** *Any linear quasigroup over the group  $S_n$  ( $n \neq 2; 6$ ) is not orthogonal to its (12)-parastrophe.*

*Proof.* The proof follows from Theorem 47 and Lemma 46. □

From Theorem 47 it follows

**Corollary 50.** [9, Theorem 17]. *For a T-quasigroup  $(Q, A)$  of the form  $A(x, y) = \varphi x + \psi y + a$  over an abelian group  $(Q, +)$  the following equivalences are fulfilled:*

- (i)  $A \perp A^{12} \iff (\varphi - \psi), (\varphi + \psi)$  are permutations of the set  $Q$ ;
- (ii)  $A \perp A^{13} \iff (\varepsilon + \varphi)$  is a permutation of the set  $Q$ ;
- (iii)  $A \perp A^{23} \iff (\varepsilon + \psi)$  is a permutation of the set  $Q$ ;
- (iv)  $A \perp A^{123} \iff (\varphi + \psi^2)$  is a permutation of the set  $Q$ ;
- (v)  $A \perp A^{132} \iff (\varphi^2 + \psi)$  is a permutation of the set  $Q$ .

**Theorem 51.** *For an alinear quasigroup  $(Q, A)$  of the form  $A(x, y) = I\varphi x + I\psi y + c$  over a group  $(Q, +)$  the following equivalences are fulfilled:*

1.  $A \perp A^{12} \iff$  the map  $(\psi^{-1} \varphi - J_t \varphi^{-1} \psi)$  is a permutation of the set  $Q$  for any  $t \in Q$ ;
2.  $A \perp A^{13} \iff$  the map  $(\varphi - J_{\psi t} J_c)$  is a permutation of the set  $Q$  for any  $t \in Q$ ;
3.  $A \perp A^{23} \iff$  the map  $(\varepsilon + I\psi J_t)$  is a permutation of the set  $Q$  for any  $t \in Q$ ;
4.  $A \perp A^{123} \iff$  the map  $(\psi^2 - \varphi J_{\psi^{-1} c})$  is a permutation of the set  $Q$ ;
5.  $A \perp A^{132} \iff$  the map  $(\psi - \varphi^2)$  is a permutation of the set  $Q$ .

*Proof.* The forms of parastrophes of quasigroup  $(Q, A)$  are given in Lemma 16.

Case 1. The proof follows from Theorem 42.

Case 2. Using Theorem 42 we have:  $A \perp A^{13}$  if and only if the maps  $\psi^{-1} \varphi - J_t \psi^{-1} \varphi J_{\varphi^{-1} c} \varphi^{-1}$  are permutation of the set  $Q$ .

We make the following transformations:  $\psi^{-1} \varphi - J_t \psi^{-1} \varphi J_{\varphi^{-1} c} \varphi^{-1} = \psi^{-1} \varphi - \psi^{-1} J_{\psi t} J_c \varphi \varphi^{-1} = \psi^{-1} \varphi - \psi^{-1} J_{\psi t} J_c = \psi^{-1} (\varphi - J_{\psi t} J_c)$ . The last maps are permutations if and only if the map  $(\varphi - J_{\psi t} J_c)$  is a permutations of the set  $Q$  for any  $t \in Q$ .

Case 3. Using Theorem 42 we have:  $A \perp A^{23}$  if and only if the maps  $\psi^{-1} \varphi - J_t \psi J_{\psi^{-1} c} J_{\psi^{-1} c} I \psi^{-1} \varphi$  are permutations of the set  $Q$ .

We simplify the last equality in the following way:

$$\psi^{-1}\varphi - J_t\psi J_{\psi^{-1}c}J_{\psi^{-1}c}I\psi^{-1}\varphi = \psi^{-1}\varphi - J_t\varphi = \psi^{-1}(\varepsilon - \psi J_t)\varphi.$$

Therefore  $A \perp A^{23}$  if and only if the maps  $(\varepsilon + I\psi J_t)$  are permutations of the set  $Q$ .

Case 4. From Theorem 29 it follows that  $A \perp A^{123}$  if and only if the map

$$I\varphi^{-1}I\psi + \psi^{-1}\varphi IJ_{\varphi^{-1}c}\varphi^{-1} \quad (27)$$

is a permutation of the set  $Q$ . We make the following transformation of expression (27):

$$\begin{aligned} I\varphi^{-1}I\psi + \psi^{-1}\varphi IJ_{\varphi^{-1}c}\varphi^{-1} &= \\ \varphi^{-1}\psi - \psi^{-1}J_c\varphi\varphi^{-1} &= \\ \varphi^{-1}\psi - \psi^{-1}J_c. \end{aligned} \quad (28)$$

We obtain:  $A \perp A^{123}$  if and only if the map  $(\varphi^{-1}\psi - \psi^{-1}J_c)$  is a permutation of the set  $Q$ .

Further we have: the map  $(\varphi^{-1}\psi - \psi^{-1}J_c)$  is a permutation of the set  $Q$  if and only if the map  $\varphi(\varphi^{-1}\psi - \psi^{-1}J_c)\psi = (\psi^2 - \varphi J_{\psi^{-1}c})$  is a permutation of the set  $Q$ .

Therefore,  $A \perp A^{123}$  if and only if the map  $(\psi^2 - \varphi J_{\psi^{-1}c})$  is a permutation of the set  $Q$ .

Case 5. From Theorem 29 we have:  $A \perp A^{132}$  if and only if the map  $(\varphi^{-1}\psi - \varphi)$  is a permutation of the set  $Q$ .

Therefore  $A \perp A^{132}$  if and only if the map  $(\varphi^{-1}\psi - \varphi) = \varphi^{-1}(\psi - \varphi^2)$  is a permutation of the set  $Q$ .  $\square$

**Corollary 52.** *Any alinear quasigroup over the group  $S_n$  ( $n \neq 2; 6$ ) is not orthogonal to its (12)-, (13)-, and (23)-parastrophe.*

*Proof.* It is possible to use Theorem 51 and Lemma 46.

We give direct proof. From Case 1 of Theorem 51 and properties of the group  $S_n$  it follows that there exists an element  $w \in S_n$  such that  $(\psi^{-1}\varphi - J_w\varphi^{-1}\psi)x = 0$  for any  $x \in S_n$ .

Cases 2 and 3 are proved in the similar way.  $\square$

**Theorem 53.** *For a left linear right alinear quasigroup  $(Q, A)$  of the form  $A(x, y) = \varphi x + I\psi y + c$  over a group  $(Q, +)$  the following equivalences are fulfilled:*

1.  $A \perp A^{12} \iff$  the map  $(\varphi^{-1}\psi - \psi^{-1}\varphi)$  is a permutation of the set  $Q$ ;
2.  $A \perp A^{13} \iff$  the map  $(\varepsilon + \varphi J_{Ic})$  is a permutation of the set  $Q$ ;
3.  $A \perp A^{23} \iff$  the map  $(J_t + \psi)$  is a permutation of the set  $Q$  for any  $t \in Q$ ;
4.  $A \perp A^{123} \iff$  the map  $(\varphi + J_{\psi Ic}\psi^2)$  is a permutation of the set  $Q$ ;
5.  $A \perp A^{132} \iff$  the map  $(\varphi^2 + IJ_k\psi)$  is a permutation of the set  $Q$  for any  $k \in Q$ .

*Proof.* The forms of parastrophes of quasigroup  $(Q, A)$  are given in Lemma 18.

Case 1. The proof follows from Theorem 31.

Case 2. Using Theorem 23 we have:  $A \perp A^{13}$  if and only if the map  $I\varphi^{-1}I\psi + \varphi J_{I\varphi^{-1}c}\varphi^{-1}\psi$  is a permutation of the set  $Q$ .

We make the following transformations:  $I\varphi^{-1}I\psi + \varphi J_{I\varphi^{-1}c}\varphi^{-1}\psi = \varphi^{-1}\psi + J_{Ic}\varphi\varphi^{-1}\psi = (\varphi^{-1} + J_{Ic})\psi$ . The last map is a permutation if and only if the map  $(\varphi^{-1} + J_{Ic}) = \varphi^{-1}(\varepsilon + \varphi J_{Ic})$  is a permutation of the set  $Q$ .

Case 3. Using Theorem 43 we have:  $A \perp A^{23}$  if and only if the maps  $J_t \varphi^{-1} \psi J_{\psi^{-1}c}^{-1} J_{\psi^{-1}c} \psi^{-1} - \varphi^{-1} I \psi$  are permutations of the set  $Q$ .

We simplify the last equality in the following way:

$$J_t \varphi^{-1} \psi J_{\psi^{-1}c}^{-1} J_{\psi^{-1}c} \psi^{-1} - \varphi^{-1} I \psi = \varphi^{-1} (I J_{\varphi t} + \psi) = \varphi^{-1} (J_{I \varphi t} + \psi).$$

Therefore  $A \perp A^{23}$  if and only if the maps  $(J_t + \psi)$  are permutations of the set  $Q$  for any  $t \in Q$ .

Case 4. From Remark 32 it follows that  $A \perp A^{123}$  if and only if the map

$$\psi^{-1} \varphi + \varphi J_{I \varphi^{-1}c} \varphi^{-1} \psi = \psi^{-1} \varphi + J_{Ic} \psi \quad (29)$$

is a permutation of the set  $Q$ . We obtain:  $A \perp A^{123}$  if and only if the map  $(\psi^{-1} \varphi + J_{Ic} \psi) = \psi^{-1}(\varphi + J_{\psi Ic} \psi^2)$  is a permutation of the set  $Q$ , i.e.  $A \perp A^{123}$  if and only if the map  $(\varphi + J_{\psi Ic} \psi^2)$  is a permutation of the set  $Q$ .

Case 5. From Theorem 33 we have:  $A \perp A^{132}$  if and only if the maps  $\psi J_{\psi^{-1}c}^{-1} J_{\psi^{-1}c} \psi^{-1} \varphi + J_{I \psi^{-1}b} \varphi^{-1} I \psi = \varphi + I J_{I \psi^{-1}b} \varphi^{-1} \psi$  are permutations of the set  $Q$  for any  $b \in Q$ . Denote the expression  $I \psi^{-1} b$  by the letter  $t$ .

Then  $A \perp A^{132}$  if and only if the maps  $\varphi + I J_t \varphi^{-1} \psi$  are permutations of the set  $Q$  for any  $t \in Q$ . But  $\varphi + I J_t \varphi^{-1} \psi = \varphi^{-1}(\varphi^2 + I J_{\varphi t} \psi)$ . Therefore  $A \perp A^{132}$  if and only if the maps  $(\varphi^2 + I J_{\varphi t} \psi)$  are permutations of the set  $Q$  for any  $t \in Q$ . Denote expression  $\varphi t$  by the letter  $k$ .

Then  $A \perp A^{132}$  if and only if the maps  $(\varphi^2 + I J_k \psi)$  are permutations of the set  $Q$  for any  $k \in Q$ .  $\square$

**Corollary 54.** *Any left linear right alinear quasigroup over the group  $S_n$  ( $n \neq 2; 6$ ) is not orthogonal to its (132)-parastrophe.*

*Proof.* The proof follows from Theorem 53 and Lemma 46. In this case we can find element  $d$  such that  $J_d \psi = \varphi^2$ .  $\square$

**Theorem 55.** *For a left alinear right linear quasigroup  $(Q, A)$  of the form  $A(x, y) = I \varphi x + \psi y + c$  over a group  $(Q, +)$  the following equivalences are fulfilled:*

1.  $A \perp A^{12} \iff$  the map  $(-\varphi^{-1} \psi + \psi^{-1} \varphi)$  is a permutation of the set  $Q$ ;
2.  $A \perp A^{13} \iff$  the map  $(\varphi + I J_{(Ib+c)})$  is a permutation of the set  $Q$  for any  $b \in Q$ ;
3.  $A \perp A^{23} \iff$  the map  $(\psi + \varepsilon)$  is a permutation of the set  $Q$ ;
4.  $A \perp A^{123} \iff$  the map  $(\psi^2 + \varphi I J_t)$  is a permutation of the set  $Q$  for any  $t \in Q$ ;
5.  $A \perp A^{132} \iff$  the map  $(\varphi^2 + \psi)$  is a permutation of the set  $Q$ .

*Proof.* The forms of parastrophes of quasigroup  $(Q, A)$  are given in Lemma 19.

Case 1. The proof follows from Theorem 34.

Case 2. Using Theorem 44 we have:  $A \perp A^{13}$  if and only if the maps  $I \psi^{-1} I \varphi + J_{I \psi^{-1}b} \psi^{-1} \varphi I J_{\varphi^{-1}c} \varphi^{-1}$  are permutations of the set  $Q$  for any  $b \in Q$ .

After simplification we have  $I \psi^{-1} I \varphi + J_{I \psi^{-1}b} \psi^{-1} \varphi I J_{\varphi^{-1}c} \varphi^{-1} = \psi^{-1} \varphi + I J_{\psi^{-1}(Ib+c)} \psi^{-1} = \psi^{-1}(\varphi + I J_{(Ib+c)})$ .

Case 3. Using Theorem 27 we have:  $A \perp A^{23}$  if and only if the map  $\psi^{-1} I \varphi - \psi \psi^{-1} \varphi = \psi^{-1} I \varphi + I \varphi = (\varepsilon + \psi^{-1}) I \varphi = (\psi + \varepsilon) \psi^{-1} I \varphi$  is a permutation of the set  $Q$ .

Finally  $A \perp A^{23}$  if and only if the map  $(\psi + \varepsilon)$  is a permutation of the set  $Q$ .

Case 4. From Theorem 33 we have:  $A \perp A^{123}$  if and only if the map

$$\begin{aligned}\varphi^{-1}\psi + J_{I\psi^{-1}b}\psi^{-1}\varphi I J_{\varphi^{-1}c}\varphi^{-1} &= \\ \varphi^{-1}\psi + I J_{I\psi^{-1}b} J_{\psi^{-1}c}\psi^{-1}\varphi\varphi^{-1} &= \\ \varphi^{-1}\psi + I J_{\psi^{-1}(Ib+c)}\psi^{-1} &= \\ \varphi^{-1}(\psi^2 + \varphi I J_{\psi^{-1}(Ib+c)})\psi^{-1} &= \end{aligned}$$

is a permutation of the set  $Q$  for any  $b \in Q$ . We denote the expression  $(\psi^{-1}(Ib + c))$  by the letter  $t$ . We obtain:  $A \perp A^{123}$  if and only if the map  $(\psi^2 + \varphi I J_t)$  is a permutation of the set  $Q$  for any  $t \in Q$ .

Case 5. From Theorem 34 we have:  $A \perp A^{132}$  if and only if the map  $\psi\psi^{-1}\varphi + \varphi^{-1}\psi = \varphi + \varphi^{-1}\psi = \varphi^{-1}(\varphi^2 + \psi)$  is a permutation of the set  $Q$ .  $\square$

**Remark 56.** Using results of Lemma 22 we can put  $c = 0$  in Theorems 51, 53, and 55.

**Corollary 57.** Any left alinear right linear quasigroup over the group  $S_n$  ( $n \neq 2; 6$ ) is not orthogonal to its (13)- and (123)-parastrophe.

*Proof.* The proof follows from Theorem 55 and Lemma 46.

Indeed, we can take into consideration that  $A \perp A^{13} \iff$  the map  $(\varphi + I J_{(Ib+c)})$  is a permutation of the set  $Q$  for any  $b \in Q$  and the fact that there exists an element  $p \in S_n$  such that  $(\varphi + I J_{(Ip+c)})x = 0$  for any  $x \in Q$ .

The second case is proved in the similar way.  $\square$

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Institute of Mathematics and  
 Computer Science  
 Academy of Sciences of Moldova  
 Academiei str. 5, MD–2028 Chişinău  
 Moldova  
 E-mail: [scerb@math.md](mailto:scerb@math.md)